

ON THE HILBERT FUNCTIONS OF SETS OF POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT. Let H_X be the trigraded Hilbert function of a set X of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We show how to extract some geometric information about X from H_X . This note generalizes a similar result of Giuffrida, Maggioni, and Ragusa about sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

1. INTRODUCTION

Let K be an algebraically closed field with $\text{char}(K) = 0$, and suppose that X is a finite set of points in the projective space \mathbb{P}^n over K . The *Hilbert function* of X is the numerical function $H_X : \mathbb{N} \rightarrow \mathbb{N}$ defined by $H_X(i) = \dim_K(R/I(X))_i$ where $R = K[x_0, \dots, x_n]$ and $I(X)$ is the homogeneous ideal associated to X . The Hilbert functions of sets of points are a well studied object, see for example [4, 5, 6, 9]. Among the questions one can ask is the question of what geometric information about the set of points, e.g., the number of collinear points in X , can be inferred from the function H_X . For work on problems of this type, we point the readers to [1, 2].

Giuffrida, Maggioni, and Ragusa [7] were among the first to consider the Hilbert function of a set of points X in a multiprojective space $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$. In this context, the associated ideal $I(X)$ is a multihomogeneous ideal in an \mathbb{N}^r -graded polynomial ring R . The Hilbert function of X is the function $H_X : \mathbb{N}^r \rightarrow \mathbb{N}$ defined by $H_X(\underline{i}) = \dim_K(R/I(X))_{\underline{i}}$. Unlike the singly graded case, our understanding of these functions is far from complete. Most notably, while there exists a classification for the Hilbert functions of sets of points in \mathbb{P}^n (see [6]), no classification is known in the multigraded situation, including $\mathbb{P}^1 \times \mathbb{P}^1$. Some known results can be found in [8, 7, 10].

In this note, we introduce some new results about the Hilbert functions of points X in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (which can be scaled to $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$; see Remark 3.6). In particular, in the spirit of [1, 2], we describe how geometric information about X is encoded into the Hilbert function H_X . Specifically, Theorem 3.1 shows how the number of points on a “line” (see Definition 2.2) in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is captured. We were inspired by a similar result of Giuffrida, Maggioni, and Ragusa [7, Theorem 2.12] for points in $\mathbb{P}^1 \times \mathbb{P}^1$. Our proof, however, uses a different approach.

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2. PRELIMINARIES

We introduce the necessary notation and basic results concerning sets of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Throughout, we use \succeq to denote the natural partial order on \mathbb{N}^3 defined by $(i, j, k) \succeq (i', j', k')$ if and only if $i \geq i'$, $j \geq j'$, and $k \geq k'$. We set $R = K[x_0, x_1, y_0, y_1, z_0, z_1]$ and induce a trigrading by setting $\deg x_i = (1, 0, 0)$, $\deg y_i = (0, 1, 0)$ and $\deg z_i = (0, 0, 1)$ for $i = 1, 2$.

Suppose that

$$P = [a_0 : a_1] \times [b_0 : b_1] \times [c_0 : c_1] \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

is a point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Associated to P is the trihomogeneous ideal given by

$$I(P) = (a_1x_0 - a_0x_1, b_1y_0 - b_0y_1, c_1z_0 - c_0z_1).$$

Given any finite set of distinct points $X = \{P_1, \dots, P_s\}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, its associated ideal is the trihomogeneous ideal $I(X) = \bigcap_{i=1}^s I(P_i)$.

We will sometimes write P as $P = A \times B \times C$, with $A, B, C \in \mathbb{P}^1$, i.e., we will use A 's for the first coordinate, and so on. We shall sometimes write the generators of $I(P)$ as $I(P) = (L_A, L_B, L_C)$ where L_A is the form of degree $(1, 0, 0)$, L_B is the form of degree $(0, 1, 0)$, and L_C is the form of degree $(0, 0, 1)$.

For each $i = 1, 2, 3$, let $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ denote the natural projection map. Consequently, $\pi_1(X) = \{A_1, \dots, A_{t_1}\}$, $\pi_2(X) = \{B_1, \dots, B_{t_2}\}$, and $\pi_3(X) = \{C_1, \dots, C_{t_3}\}$ denote the sets of distinct first, second, and third coordinates of X respectively. We let $t_1 = |\pi_1(X)|$, $t_2 = |\pi_2(X)|$, and $t_3 = |\pi_3(X)|$.

Definition 2.1. The *Hilbert function* of X , denoted H_X , is the function $H_X : \mathbb{N}^3 \rightarrow \mathbb{N}$ defined by $H_X(i, j, k) = \dim_K R_{i,j,k} - \dim_K I(X)_{i,j,k}$.

Note that $\dim_K R_{i,j,k} = (i+1)(j+1)(k+1)$ since there are $i+1$ ways to make a monomial of degree i in the variables x_0 and x_1 , $j+1$ ways to make a monomial of degree j in the y_i s, and $k+1$ ways to make a monomial of degree k in the z_i s.

Our goal is to show how H_X captures geometric information about the number of points lying on linear subvarieties of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We give a more precise definition.

Definition 2.2. Let $L \in R_{1,0,0}$ and $L' \in R_{0,1,0}$. We call the variety \mathcal{L} in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the ideal $(L, L') \subseteq R$ a *line of type* $(1, 1, 0)$.

Remark 2.3. We will focus only on lines of type $(1, 1, 0)$, although similar results can be proved for lines of type $(1, 0, 1)$ and $(0, 1, 1)$, which are defined in an analogous manner.

Remark 2.4. We add a few comments about how to interpret the geometry. One can construct an embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into a projective space; in particular, using Segre's embedding $\mathbb{T} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$ using the sheaf $\mathcal{O}_{\mathbb{T}}(1, 1, 1)$. We can easily check that the ideal of the image \mathbb{T}' of \mathbb{T} under the embedding is:

$$\begin{aligned} I(\mathbb{T}') = & (u_0u_7 - u_1u_6, u_0u_7 - u_2u_5, u_0u_7 - u_3u_4, u_0u_3 - u_1u_2, u_4u_7 - u_5u_6, \\ & u_0u_5 - u_1u_4, u_2u_7 - u_3u_6, u_0u_6 - u_2u_4, u_1u_7 - u_3u_5) \end{aligned}$$

in $K[u_0, \dots, u_7]$. We can view $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as ruled 3-fold. The lines of type $(1, 1, 0)$, respectively $(1, 0, 1)$ and $(0, 1, 1)$, can be viewed as rulings on this surface. Our goal is to count the number of points on these rulings using the Hilbert function.

Alternatively, if we fix a degree $(1, 0, 0)$ form L , we can view this as fixing a divisor of type $(1, 0, 0)$. Fixing this linear equation is equivalent to fixing a point A in the first copy of \mathbb{P}^1 . We are then looking at points in the set

$$S_A = \{A \times B \times C \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid B \times C \in \mathbb{P}^1 \times \mathbb{P}^1\} \cong \mathbb{P}^1 \times \mathbb{P}^1$$

which is also isomorphic to the ruled quadric surface in \mathbb{P}^3 . So, when we form a line of type $(1, 1, 0)$, we can view it as fixing a point in a \mathbb{P}^1 and a line in a $\mathbb{P}^1 \times \mathbb{P}^1$, or equivalently, a ruling on the ruled quadric surface Q , and then we are counting the number of points on this ruling on the quadric Q . Note, by abuse of notation, the divisor of type $(1, 0, 0)$ can be called a plane of type $(1, 0, 0)$, and similarly for divisor of type $(0, 1, 0)$. So, the intersection of these two planes results in a line of type $(1, 1, 0)$.

We now provide a number of lemmas that shall be required for our main result.

Lemma 2.5. *Let X be a finite set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Fix an integer $k \geq 0$. Then*

$$H_X(t_1 - 1, t_2 - 1, k) = H_X(i, j, k) \text{ for all } (i, j, k) \succeq (t_1 - 1, t_2 - 1, k).$$

Proof. Let L be a form of degree $(1, 0, 0)$ such that no point of X lies on L . Let L' be a form of degree $(0, 1, 0)$ such that no point of X lies on L' . We observe that L , respectively L' , is a nonzerodivisor on $R/I(X)$. Thus the multiplication map

$$\times \overline{L} : (R/I(X))_{i,j,k} \rightarrow (R/I(X))_{i+1,j,k}$$

is injective for all (i, j, k) . A similar result holds using L' . Thus

$$H_X(t_1 - 1, t_2 - 1, k) \leq H_X(t_1, t_2 - 1, k) \leq \dots \leq H_X(i, t_2 - 1, k) \text{ for } i \geq t_1 - 1$$

and

$$H_X(i, t_2 - 1, k) \leq H_X(i, t_2, k) \leq \dots \leq H_X(i, j, k) \text{ for } j \geq t_2 - 1.$$

In addition, we have the short exact sequence

$$0 \longrightarrow R/I(X)(-1, 0, 0) \xrightarrow{\times \overline{L}} R/I(X) \longrightarrow R/(I(X), L) \longrightarrow 0.$$

Now $H_X(t_1 - 1, 0, 0) = H_X(t_1, 0, 0) = |\pi_1(X)|$ because $\bigoplus_{i \in \mathbb{N}} I(X)_{i,0,0} \cong I(\pi_1(X)) \subseteq K[x_0, x_1]$, i.e., the ideal of the t_1 points $\pi_1(X)$ in \mathbb{P}^1 . So, the short exact sequence implies that $(I(X), L)_{i,0,0} = R_{i,0,0}$ for all $i \geq t_1$. But this means that the multiplication map $\times \overline{L}$ is also surjective for (i, j, k) with $i \geq t_1 - 1$. Thus $H_X(t_1 - 1, t_2 - 1, k) = H_X(t_1, t_2 - 1, k) = \dots = H_X(i, t_2 - 1, k)$.

We now apply a similar argument with L' to show that $H_X(i, t_2 - 1, k) = H_X(i, t_2, k) = \dots = H_X(i, j, k)$. \square

Lemma 2.6. *Let \mathcal{L} be a line of type $(1, 1, 0)$, and suppose that X is a finite set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $X \subseteq \mathcal{L}$. If $|X| = s$, then*

$$H_X(i, j, k) = \min\{k + 1, s\} \text{ for all } (i, j, k) \in \mathbb{N}^3.$$

Proof. Because $X \subseteq \mathcal{L}$, the set X has the form

$$X = \{A \times B \times C_1, A \times B \times C_2, \dots, A \times B \times C_s\}$$

for some $A, B, C_i \in \mathbb{P}^1$. After a change of coordinates, we can assume that $A = [1 : 0]$ and $B = [1 : 0]$. Thus

$$I(X) = \bigcap_{i=1}^s (x_1, y_1, L_{C_i}) = (x_1, y_1, L_{C_1} L_{C_2} \cdots L_{C_s}).$$

If we set $L = L_{C_1} \cdots L_{C_s}$, then $R/I(X) \cong K[x_0, y_0, z_0, z_1]/(L)$; here, we are viewing L as an element of $K[x_0, y_0, z_0, z_1]$. If $S = K[x_0, y_0, z_0, z_1]$ is the \mathbb{N}^3 -graded ring with $\deg x_0 = (1, 0, 0)$, $\deg y_0 = (0, 1, 0)$ and $\deg z_i = (0, 0, 1)$, then the result follows by using the short exact sequence

$$0 \longrightarrow S(0, 0, -s) \xrightarrow{\times L} S \longrightarrow S/(L) \longrightarrow 0$$

to compute H_X and the fact that $\dim_K S_{i,j,k} = k + 1$ for all $(i, j) \in \mathbb{N}^2$. \square

Lemma 2.7. *Let X be a finite set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and suppose that \mathcal{L} is a line of type $(1, 1, 0)$ that intersects X , but $X \not\subseteq \mathcal{L}$. Set $X_2 = X \cap \mathcal{L}$ and $X_1 = X \setminus X_2$. Then*

$$R_{t_1-1, t_2-1, 0} = (I(X_1) + I(X_2))_{t_1-1, t_2-1, 0}.$$

Consequently,

$$H_{R/(I(X_1)+I(X_2))}(t_1-1, t_2-1, k) = 0 \text{ for all } k \geq 0.$$

Proof. The second statement follows from the first since $(I(X_1) + I(X_2))_{t_1-1, t_2-1, k} = R_{t_1-1, t_2-1, k}$ for all $k \geq 0$ if the first statement holds.

After a change of coordinates, we can assume that \mathcal{L} is the line defined by x_1 and y_1 , i.e., every point on X_2 has the form $[1 : 0] \times [1 : 0] \times C$ for some $C \in \mathbb{P}^1$. As in the proof of Lemma 2.6, we can take $I(X_2) = (x_1, y_1, L)$ where L is some homogeneous polynomial only in the variables z_0 and z_1 . Now for any $(i, j) \in \mathbb{N}^2$, $I(X_2)_{i,j,0}$ contains all the monomials of degree $(i, j, 0)$ except $x_0^i y_0^j$.

Now $\pi_1(X) = \{A_1, \dots, A_{t_1-1}, [1 : 0]\}$ and $\pi_2(X) = \{B_1, \dots, B_{t_2-1}, [1 : 0]\}$. Note that because $X \not\subseteq \mathcal{L}$, either $t_1 \geq 2$ or $t_2 \geq 2$. Let L_{A_i} be the form of degree $(1, 0, 0)$ that vanishes at points of the form $A_i \times B \times C$ where $B, C \in \mathbb{P}^1$ and let L_{B_j} be the form of degree $(0, 1, 0)$ that vanishes at all points of the form $A \times B_j \times C$ with $A, C \in \mathbb{P}^1$. Then $H = L_{A_1} \cdots L_{A_{t_1-1}} L_{B_1} \cdots L_{B_{t_2-1}}$ is a degree $(t_1-1, t_2-1, 0)$ form that vanishes at all the points of X_1 with $t_1-1 > 0$ or $t_2-1 > 0$, that is, $(t_1-1, t_2-1, 0) \neq (0, 0, 0)$. Thus $H \in I(X_1)_{t_1-1, t_2-1, 0}$ and H is not a constant. Furthermore, since H does not vanish at any of the points in X_2 , neither x_1 nor y_1 divides H . So H must have the form $H = cx_0^{t_1-1} y_0^{t_2-1} + H'$ where $c \neq 0$.

Since $H \in I(X_1)_{t_1-1, t_2-1, 0}$ and since all the monomials of degree $(t_1-1, t_2-1, 0)$ except $x_0^{t_1-1} y_0^{t_2-1}$ belong to $I(X_2)_{t_1-1, t_2-1, 0}$, we must have all the monomials of degree $(t_1-1, t_2-1, 0)$ belonging to $(I(X_1) + I(X_2))_{t_1-1, t_2-1, 0}$, from which our conclusion follows. \square

3. MAIN RESULT

In this section we prove our main result which explains how to extract geometric information about X from H_X . In particular, we are able to determine information about the number of points on lines of type $(1, 1, 0)$.

We make use of the following notation. If X is a finite set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, for each integer $i \geq 1$, let

$r_i(X) :=$ the number of lines of type $(1, 1, 0)$ that contain exactly i points of X .

Because X is finite, $r_i(X) = 0$ for all but a finite number of i . Given H_X , we set

$$d_{i,j,k} := H_X(i, j, k) - H_X(i, j, k-1) \text{ for all } (i, j, k) \in \mathbb{N}^3$$

where $H_X(a, b, c) = 0$ if $(a, b, c) \not\geq (0, 0, 0)$. We shall be interested in the sequence $\{d_{t_1-1, t_2-1, k} - d_{t_1-1, t_2-1, k+1}\}_{k \in \mathbb{N}}$. It is not obvious that this is a sequence of nonnegative numbers. However, even though one can prove this directly from the properties of the Hilbert function, this fact will be an immediate consequence of our main theorem, stated below, which shows how geometric information is encoded into the sequence.

Theorem 3.1. *Let X be a finite set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $t_1 = |\pi_1(X)|$ and $t_2 = |\pi_2(X)|$. For every $k \geq 0$,*

$$\begin{aligned} r_{k+1}(X) &= d_{t_1-1, t_2-1, k} - d_{t_1-1, t_2-1, k+1} \\ &= 2H_X(t_1-1, t_2-1, k) - H_X(t_1-1, t_2-1, k-1) - H_X(t_1-1, t_2-1, k+1). \end{aligned}$$

Remark 3.2. As we see in the above theorem, $r_{k+1}(X)$ can be computed directly from the Hilbert function H_X or through the $d_{i,j,k}$ s. We have presented the formula for $r_{k+1}(X)$ using the $d_{i,j,k}$ s because it is reminiscent of the formulas found in [7, Theorem 2.12] for points in $\mathbb{P}^1 \times \mathbb{P}^1$ which inspired this result.

Let us illustrate this result before turning to its proof.

Example 3.3. We show how to go from the Hilbert function to information about the set of points. Suppose that we are given the following trigraded Hilbert function for a set of points X . Below, H_X is written as a collection of infinite matrices, where the initial row and column are indexed with 0 as opposed to 1. You should view these matrices as the “layers” of the box matrix that make up H_X .

$$H_X(i, j, 0) = \begin{bmatrix} 1 & 2 & 3 & 3 & \cdots \\ 2 & 4 & 5 & 5 & \cdots \\ 3 & 5 & 6 & 6 & \cdots \\ 3 & 5 & 6 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad H_X(i, j, 1) = \begin{bmatrix} 2 & 4 & 5 & 5 & \cdots \\ 4 & 7 & 8 & 8 & \cdots \\ 5 & 8 & 9 & 9 & \cdots \\ 5 & 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad H_X(i, j, 2) = \begin{bmatrix} 3 & 6 & 7 & 7 & \cdots \\ 5 & 9 & 10 & 10 & \cdots \\ 6 & 10 & 11 & 11 & \cdots \\ 6 & 10 & 11 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and $H_X(i, j, k) = H_X(i, j, 2)$ for all $k \geq 2$.

From this Hilbert function, we can see that $t_1 = t_2 = t_3 = 3$. We can use Theorem 3.1 to determine the number of points on each line of type $(1, 1, 0)$. We need the values of $H_X(2, 2, l)$ for all $l \in \mathbb{N}$. From H_X above, we have

$$H_X(2, 2, 0) = 6, \quad H_X(2, 2, 1) = 9, \quad \text{and} \quad H_X(2, 2, l) = 11 \text{ for all } l \geq 2.$$

So, $d_{2,2,0} = 6$, $d_{2,2,1} = 3$, $d_{2,2,2} = 2$, and $d_{2,2,l} = 0$ otherwise. By Theorem 3.1, we then have $d_{2,2,0} - d_{2,2,1} = 3$ lines of type $(1, 1, 0)$ that contain exactly one point of X , $d_{2,2,1} - d_{2,2,2} = 1$ lines of type $(1, 1, 0)$ that contain exactly two points of X , and $d_{2,2,2} - d_{2,2,3} = 2$ lines of type $(1, 1, 0)$ that contain exactly three points of X .

Indeed, H_X is the Hilbert function of the set of points X constructed as below. Let $A_i := [1 : i] \in \mathbb{P}^1$, $B_i := [1 : i] \in \mathbb{P}^1$ and $C_i := [1 : i] \in \mathbb{P}^1$. Then X is the following scheme

$$X := \{P_{111}, P_{112}, P_{113}, P_{121}, P_{122}, P_{123}, P_{212}, P_{211}, P_{311}, P_{221}, P_{131}\}$$

where $P_{ijk} = A_i \times B_j \times C_k$. If L_{A_i} denotes the $(1, 0, 0)$ form that vanishes at all points of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose first coordinate is A_i , and similarly, if L_{B_j} denotes the $(0, 1, 0)$ form that vanishes at all points whose second coordinate is B_j , then (L_{A_1}, L_{B_1}) and (L_{A_1}, L_{B_2}) are the two lines of type $(1, 1, 0)$ that contain exactly three points of X .

Theorem 3.1 will follow from the next result.

Theorem 3.4. *Let X be a finite set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $t_1 = |\pi_1(X)|$ and $t_2 = |\pi_2(X)|$. Then for all $k \geq 0$,*

$$H_X(t_1 - 1, t_2 - 1, k) = \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X) \right).$$

Proof. We do induction on the number of lines of type $(1, 1, 0)$ that intersect with X . For the base case, suppose that there is only one line \mathcal{L} of type $(1, 1, 0)$ that intersects X . In other words, X is contained on \mathcal{L} , whence $t_1 = t_2 = 1$. If $|X| = s$, then $r_s(X) = 1$ and $r_{s'}(X) = 0$ for all $s' \neq s$. By Lemma 2.6,

$$H_X(0, 0, k) = \min\{k + 1, s\} \text{ for all } k \geq 0.$$

On the other hand, note that $\sum_{n=k+1}^{\infty} r_n(X) = 1$ if $k + 1 \leq s$ and zero otherwise. This implies that

$$\sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X) \right) = \min\{k + 1, s\}.$$

So the formulas agree.

For the induction step, suppose that there are $t > 1$ lines $\mathcal{L}_1, \dots, \mathcal{L}_t$ of type $(1, 1, 0)$ that intersect X . Set $X_2 = X \cap \mathcal{L}_t$ and $X_1 = X \setminus X_2$. We now consider the following short exact sequence

$$0 \longrightarrow R/(I(X_1) \cap I(X_2)) \longrightarrow R/I(X_1) \oplus R/I(X_2) \longrightarrow R/(I(X_1) + I(X_2)) \longrightarrow 0.$$

From this short exact sequence we have

$$(3.1) \quad H_X(t_1 - 1, t_2 - 1, k) = H_{X_1}(t_1 - 1, t_2 - 1, k) + H_{X_2}(t_1 - 1, t_2 - 1, k) - H_{R/(I(X_1) + I(X_2))}(t_1 - 1, t_2 - 1, k)$$

for all $k \geq 0$.

Since there are only $t - 1$ lines of type $(1, 1, 0)$ that intersect with X_1 and only one line of type $(1, 1, 0)$ that intersects with X_2 , the induction hypothesis gives

$$H_{X_1}(t_1(X_1) - 1, t_2(X_1) - 1, k) = \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X_1) \right)$$

and

$$H_{X_2}(t_1(X_2) - 1, t_2(X_2) - 1, k) = \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X_2) \right)$$

where $t_i(X_j) = |\pi_i(X_j)|$ for $i = 1, 2$ and $j = 1, 2$. Because $t_1 \geq t_1(X_1), t_1(X_2)$ and $t_2 \geq t_2(X_1), t_2(X_2)$, Lemma 2.5 then implies that

$$\begin{aligned} H_{X_1}(t_1 - 1, t_2 - 1, k) &= \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X_1) \right) \text{ and} \\ H_{X_2}(t_1 - 1, t_2 - 1, k) &= \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X_2) \right). \end{aligned}$$

By Lemma 2.7, $H_{R/(I(X_1)+I(X_2))}(t_1 - 1, t_2 - 1, k) = 0$ for all $k \geq 0$. Furthermore, $r_n(X) = r_n(X_1) + r_n(X_2)$ for all $n \geq 1$. Thus, (3.1) gives the desired formula:

$$\begin{aligned} H_X(t_1 - 1, t_2 - 1, k) &= \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X_1) \right) + \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X_2) \right) \\ &= \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} (r_n(X_1) + r_n(X_2)) \right) = \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X) \right). \end{aligned}$$

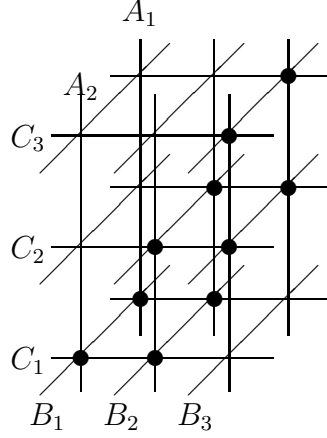
□

Example 3.5. In this example, we show how Theorem 3.4 can be used to determine values of H_X directly from information about the set of points X .

Let $A_i := [1 : i] \in \mathbb{P}^1$, $B_i := [1 : i] \in \mathbb{P}^1$ and $C_i := [1 : i] \in \mathbb{P}^1$, and suppose that X is the following scheme

$$X := \{P_{111}, P_{121}, P_{122}, P_{132}, P_{133}, P_{211}, P_{221}, P_{222}, P_{232}, P_{233}\}$$

where $P_{ijk} = A_i \times B_j \times C_k$. We can visualize this set of points as follows:



Note that in the above picture, the vertical lines represent the lines of type $(1, 1, 0)$.

From X , we can compute $r_i(X)$ for all i . First, let L_{A_i} denote the $(1, 0, 0)$ form that vanishes at all points of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose first coordinate is A_i , and similarly, let L_{B_j} denote the $(0, 1, 0)$ form that vanishes at all points whose second coordinate is B_j . Then the line of type $(1, 1, 0)$ given by (L_{A_1}, L_{B_1}) contains one point of X , the line of type $(1, 1, 0)$ given by (L_{A_1}, L_{B_2}) contains two points of X , the line of type $(1, 1, 0)$ given by (L_{A_1}, L_{B_3}) contains two points, the line of type $(1, 1, 0)$ given by (L_{A_2}, L_{B_1}) contains one point, the line of type $(1, 1, 0)$ given by (L_{A_2}, L_{B_2}) contains two points, and the line of type $(1, 1, 0)$ given by (L_{A_2}, L_{B_3}) contains two points.

From this data, and from X , we have $t_1 = 2$ and $t_2 = 3$, and $r_1(X) = 2$, $r_2(X) = 4$, and $r_i(X) = 0$ for $i \geq 3$. So,

$$\begin{aligned} H_X(1, 2, 0) &= r_1(X) + r_2(X) = 6 \\ H_X(1, 2, 1) &= r_1(X) + r_2(X) + r_2(X) = 10 \\ H_X(1, 2, 2) &= r_1(X) + r_2(X) + r_2(X) = 10 = H_X(1, 2, l) \text{ for } l \geq 2. \end{aligned}$$

Note that by Lemma 2.5, we have actually computed an infinite number of values of H_X . For example, for all $(i, j, 2) \succeq (1, 2, 2)$, we have $H_X(i, j, 2) = H_X(1, 2, 2) = 10$.

As a reminder, a similar result to Theorem 3.4 also holds for the number of points on a line of type $(1, 0, 1)$ and $(0, 1, 1)$. Since $t_3 = 3$, we can also compute the values of $H_X(t_1 - 1, k, t_3 - 1)$ and $H_X(k, t_2 - 1, t_3 - 1)$ for all $k \geq 0$. We omit the details.

Proof of Theorem 3.1. For any $k \geq 0$,

$$\begin{aligned} d_{t_1-1, t_2-1, k} &= H_X(t_1 - 1, t_2 - 1, k) - H_X(t_1 - 1, t_2 - 1, k - 1) \\ &= \sum_{m=1}^{k+1} \left(\sum_{n=k+1}^{\infty} r_n(X) \right) - \sum_{m=1}^k \left(\sum_{n=k}^{\infty} r_n(X) \right) = \sum_{n=k+1}^{\infty} r_n(X). \end{aligned}$$

Thus

$$d_{t_1-1, t_2-1, k} - d_{t_1-1, t_2-1, k+1} = \sum_{n=k+1}^{\infty} r_n(X) - \sum_{n=k+2}^{\infty} r_n(X) = r_{k+1}(X).$$

□

Remark 3.6. We point out that although Theorem 3.1 is only stated for points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, one can adapt the proofs to scale this result to $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (r copies). In particular, one defines a *line of type* $(\underbrace{1, \dots, 1}_r, 0)$ in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ as the variety defined

by linear forms of degree e_1, \dots, e_{r-1} where e_i is the standard basis vector of \mathbb{N}^r . If $t_i = |\pi_i(X)|$ is the number of distinct i -th coordinates that appear in X , then the number of points on a lines of type $(1, \dots, 1, 0)$ is encoded into the sequence

$$\{H_X(t_1 - 1, \dots, t_{r-1} - 1, k)\}_{k \in \mathbb{N}}.$$

Our result can also be seen as a generalization of [7, Theorem 2.12] which determined the geometric information encoded into the sequence $\{H_X(t_1 - 1, k)\}_{k \in \mathbb{N}}$ when X is a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $t_1 = |X|$.

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REFERENCES

- [1] A. Bigatti, A.V. Geramita, J. Migliore. Geometric consequences of extremal behavior in a theorem of Macaulay. *Trans. Amer. Math. Soc.* **346** (1994) 203–235.
- [2] L. Chiantini, J. Migliore. Almost maximal growth of the Hilbert function. Preprint (2014) [arXiv:1403.1409](https://arxiv.org/abs/1403.1409)
- [3] CoCoATeam. CoCoA: A system for doing Computations in Commutative Algebra. Available at <http://cocoa.dima.unige.it>
- [4] A.V. Geramita, D. Gregory, L.G. Roberts. Monomial ideals and points in projective space. *J. Pure Appl. Algebra* **40** (1986) 33–62.
- [5] A.V. Geramita, T. Harima, Y.S. Shin. An alternative to the Hilbert function for the ideal of a finite set of points in \mathbb{P}^n . *Illinois J. Math.* **45** (2001) 1–23.
- [6] A.V. Geramita, P. Maroscia, L.G. Roberts. The Hilbert function of a reduced k -algebra. *J. London Math. Soc.* **28** (1983) 443–452.
- [7] S. Giffuffrida, R. Maggioni, A. Ragusa. On the postulation of 0-dimensional subschemes on a smooth quadric. *Pacific J. Math.* **155** (1992) 251–282.
- [8] E. Guardo. Fat point schemes on a smooth quadric. *J. Pure Appl. Algebra* **162** (2001) 183–208.
- [9] R. Maggioni, A. Ragusa. The Hilbert function of generic plane sections of curves of \mathbb{P}^3 . *Invent. Math.* **91** (1988) 253–258.
- [10] A. Van Tuyl. The border of the Hilbert function of a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. *J. Pure Appl. Algebra* **176** (2002) 223–247.

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